MATHEMATICAL UNDERSTANDING FOR TEACHING SECONDARY MATHEMATICS WITH CAS

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Goal: produce a framework for secondary teachers’ understanding of mathematics

- Base the framework on actual classroom interactions
- Elaborate mathematical understanding for teaching at the secondary level
Start with the mathematics classroom. Identify events that are mathematical opportunities

Components:

- Prompt---the classroom event
- Mathematical Foci ---what mathematics could the teacher productively use?
- Capsule statement of each Mathematical Focus
- Commentary
- Post-Commentary
Major Faculty involved:

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Steps in developing the framework

- **Identify mathematical opportunities** from observing secondary classrooms.
- **Determine and describe mathematical foci** -- mathematics a teacher could productively use in each situation.
- Based on the mathematical foci, **determine perspectives** that capture the mathematics teachers could use in these situations.
- **Develop the subcategories** within each perspective.
- **Solicit feedback** from mathematicians, mathematics educators, and secondary mathematics teachers.
FRAMEWORK FOR MATHEMATICAL UNDERSTANDING FOR SECONDARY TEACHING

A synthesis of mathematical elaborations from more than 50 Situations

MP  Mathematical Proficiency
MA  Mathematical Activity
MC  Mathematical Context
FRAMEWORK FOR MATHEMATICAL UNDERSTANDING FOR SECONDARY TEACHING

Mathematical Proficiency
- Conceptual Understanding
- Procedural Fluency
- Strategic Competence
- Adaptive Reasoning
- Productive Disposition
- Historical & Cultural Knowledge

Mathematical Activity
- Mathematical Noticing
- Mathematical Reasoning
- Mathematical Creating

Mathematical Context of Teaching
- Probe Mathematical Ideas
- Understand Students’ Mathematical Thinking
- Know and Use Curriculum
- Assess Mathematical Knowledge of Learners
- Reflect on the Mathematics of Practice
A student was asked to produce a function that had certain given characteristics. One of those characteristics was that the function should be undefined for values less than 5. Another characteristic was that the range of the function should contain only non-negative values. In the process, the student defined \( f(x) = \sqrt{x - 5} \) and then evaluated \( f(-10) \) using his CAS calculator. The calculator displayed a result of 3.872983346. He looked at the calculator screen and whispered, “How can that be?”
MATHEMATICAL FOCUS 1

Complex numbers can be represented as points on the complex plane.
The absolute value of a complex number, \( z = x + yi \), is the number’s distance from the origin. This distance is called the modulus or norm and is computed by \( |z| = \sqrt{x^2 + y^2} \).
MATHEMATICAL FOCUS 3

In the complex plane, there are infinitely many solutions to any linear absolute value equation other than $|z| = 0$, and the graph of these solutions forms a circle.

\[
|x| = 3 \\
\Rightarrow |a + bi| = 3 \\
\Rightarrow \sqrt{a^2 + b^2} = 3 \\
\Rightarrow a^2 + b^2 = 9
\]
In the complex plane, there are infinitely many solutions to any linear absolute value equation other than $|z| = 0$, and the graph of these solutions forms a circle.

**Real number solutions:**

$|3x + 1| = 5 \Rightarrow x = \frac{4}{3} \text{ or } x = -2$

**Complex number solutions:**

$|3x + 1| = 5$

$\Rightarrow |3(a + bi) + 1| = 5$

$\Rightarrow |3a + 1 + 3bi| = 5$

$\Rightarrow \sqrt{(3a + 1)^2 + 3b^2} = 5$

$\Rightarrow \sqrt{9a^2 + 6a + 1 + 9b^2} = 5$

$\Rightarrow \cdots$

$\Rightarrow \left( a + \frac{1}{3} \right)^2 + b^2 = \left( \frac{5}{3} \right)^2$
MATHEMATICAL FOCUS 4

A composite function with the same domain and codomain may be composed of functions with different domains and codomains.
Knowing that a Computer Algebra System (CAS) had commands such as \texttt{cfactor} and \texttt{csolve} to factor complex number expressions and solve complex number equations, a teacher was curious about what would happen if she entered $\sqrt{i}$.

The result was $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

She wondered why a CAS would give a result such as that.

Johnson, Kararunakaran, Fox, McClintock
Solving the equation $x^2 = i$ where $c = a + b \, i$ and verifying the solution to the equation provides a representation of the imaginary number $i$. 
Powers of $i$ can be related to rotations involving the unit circle on the complex plane.
MATHEMATICAL FOCUS 3

By using Euler’s formula, the connection between the trigonometric representation of any complex number and the square root of the imaginary number i, is made more explicit.
The value of the square root of the imaginary number $i$ can be determined by related this value to cyclic groups.
What are some other examples of prompts arising in the context of CAS use?