XM: Introduction

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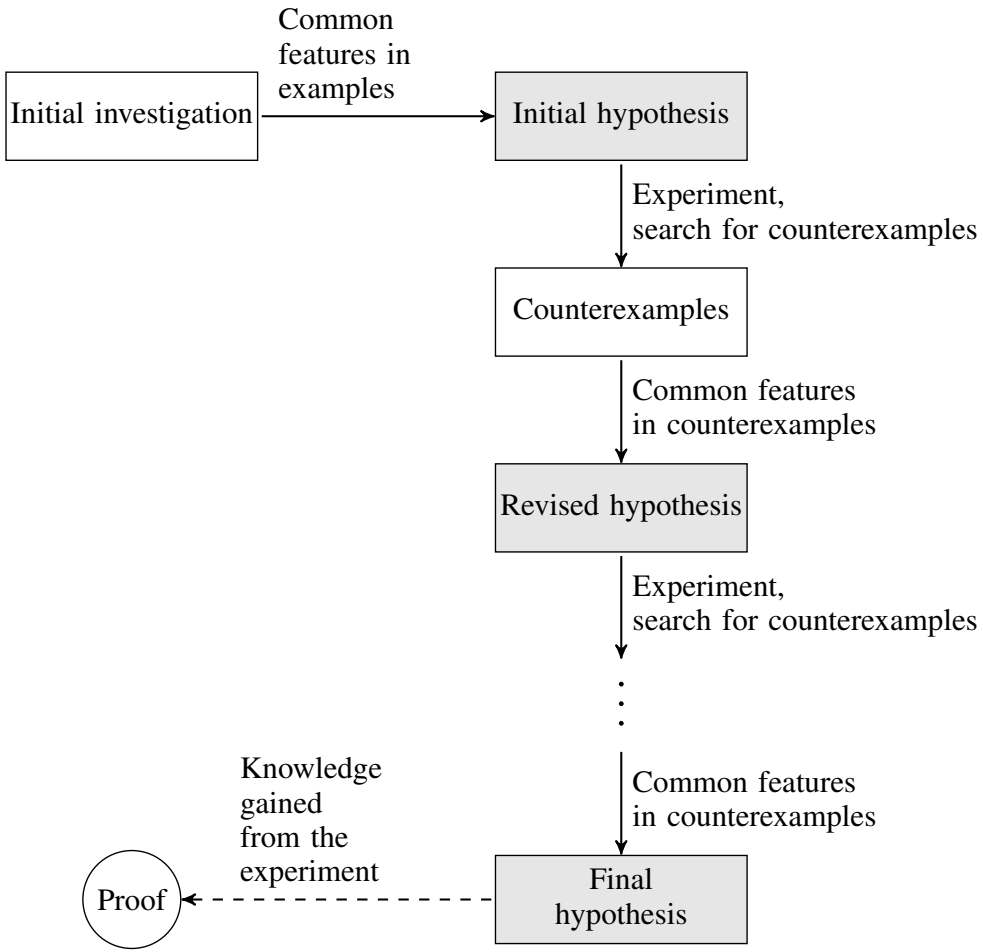
Outline


Initial investigation
Common features in examples
Initial hypothesis
Experiment, search for counterexamples
Counterexamples
Common features in counterexamples
Revised hypothesis
Experiment, search for counterexamples
Common features in counterexamples
Final hypothesis
Knowledge gained from the experiment
Proof

Experimental mathematics

The systematic investigation of concrete examples of a mathematical structure in the search for conjectures about its properties (using computers).

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Typical output from an experimental investigation:

- A counterexample
- A reference to the mathematical literature
- A “hint” for a proof
- A conjecture

Conjecture *[L. Euler 1769]*

$$x_1^n + x_2^n + \cdots + x_m^n = y^n$$

can only have solutions $x_1, x_2, \ldots, x_m, y \in \mathbb{N}$ when $n \geqslant m$ for $m \geqslant 2$; i.e., for a sum of nth powers to itself be an nth power it must have either only one or at least n summands.

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Euler had essentially proved that

$$x_1^3 + x_2^3 = y^3$$

did not have any such solutions, cf. Fermat's last theorem.

COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

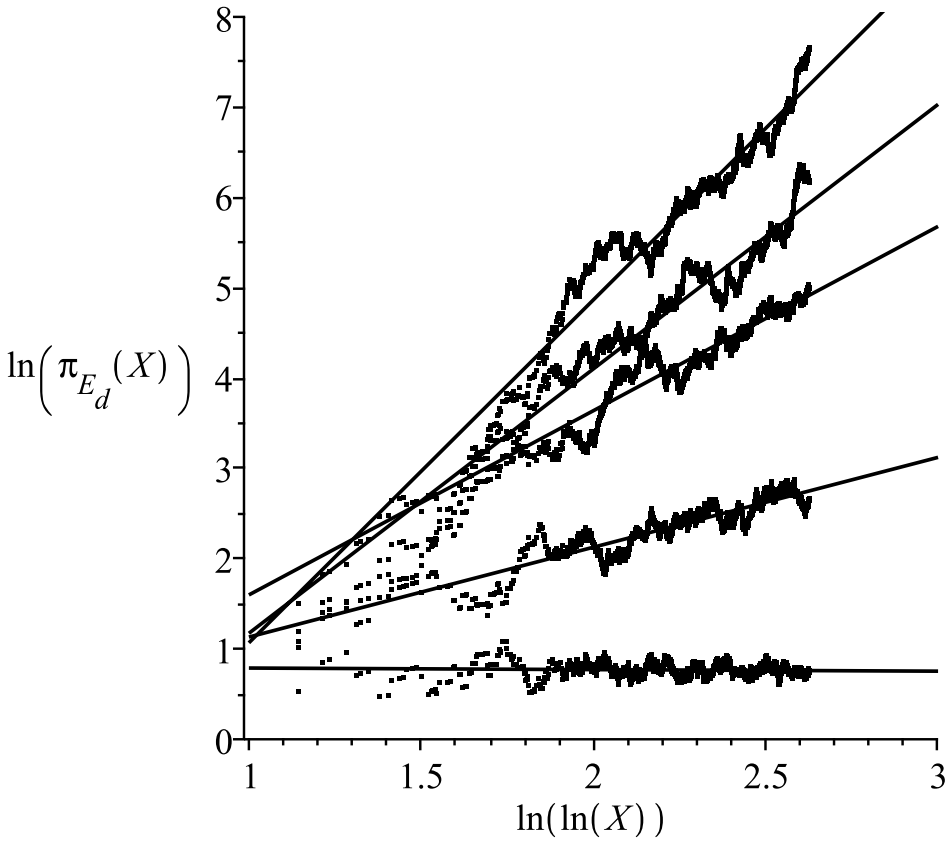
A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [**1**] that at least n nth powers are required to sum to an nth power, $n > 2$.

Reference

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.



Birch & Swinnerton-Dyer: Data for the elliptic curves $y^2 = x^3 - d^2x$ at $d = 1, 5, 34, 1254, 29274$.

Consider an elliptic curve

$$E : y^2 = x^3 + ax + b \quad , \quad a, b \in \mathbb{Z}$$

with discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$. For every prime p not dividing Δ, consider

$$N_p = 1 + \left|\{0 \leqslant x, y \leqslant p-1 \mid y^2 \equiv x^3 + ax + b \mod p\}\right|.$$

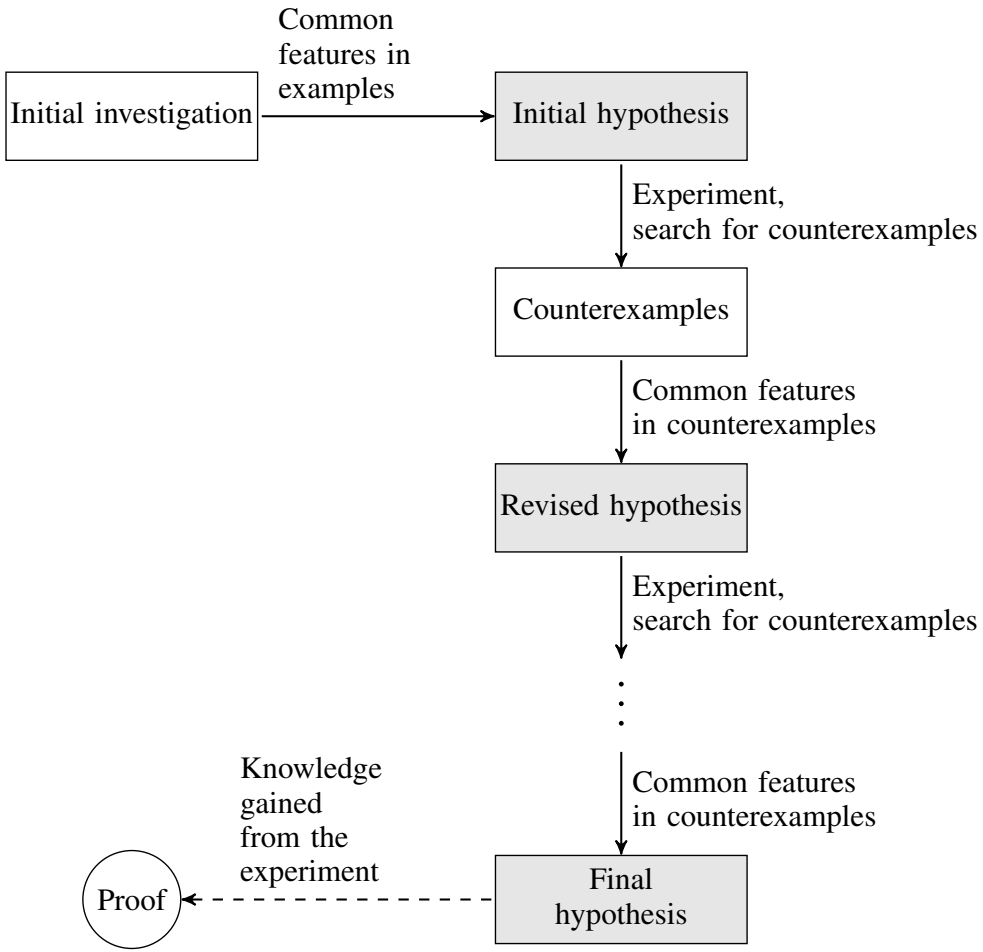
Birch and Swinnerton-Dyer got the idea that N_p should be related to the *rank* of E, and over a period of five years in the late 1950's and early 1960's they computed

$$\pi_E(X) = \prod_{p \leqslant X, p \nmid \Delta} \frac{N_p}{p}$$

for many different values of X for a number of elliptic curves with known ranks, leading to the conjecture that

$$\pi_E(X) \to C \ln(X)^{\mathrm{rank}(E)} \text{ for } X \to \infty$$

for some C that depends only on E.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$$
\begin{aligned}
2^1 + 1 &= 3 \text{ is a prime} \\
2^2 + 1 &= 5 \text{ is a prime} \\
2^4 + 1 &= 17 \text{ is a prime} \\
2^8 + 1 &= 257 \text{ is a prime} \\
2^{16} + 1 &= 65537 \text{ is a prime}
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Example *[L. Euler 1732]*

$F_5 = 4294967297 = 641 \cdot 6700417$

Unter	gittes Primzahlen	Integral $\int \frac{dn}{\log n}$	Differ	Ihre Formel	Abweich.
500 000	41 556	41606,4	+50,4	415969	+40,9
1000 000	78 501	78627,5	+126,5	786727	+171,7
1500 000	114 112	114263,1	+151,1	114374,0	+264,0
2000 000	148883	149054,8	+171,8	149233,0	+350,0
2500 000	183016	183245,0	+229,0	183495,1	+479,1
3000 000	216745	216970,6	+225,6	217308,5	+563,6

Dass Legendre sich auch mit diesem Gegenstande beschäftigt hat, war mir nicht bekannt; auf Veranlassung Ihres Briefes habe ich in seiner Theorie des Nombres nachgesehen, und in der zweiten Ausgabe einige darauf bezügliche Seiten gefunden, die ich früher übersehen (oder seitdem vergessen) haben muß. Legendre gebraucht die Formel

$$\frac{n}{\log n - A}$$

Conjecture *[C.F. Gauss, 1792–93]*

With $\pi(x)$ the number of primes less than x,

$$\pi(x) \approx \int_2^x \frac{du}{\ln u} =: \mathrm{Li}(x)$$

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The prime number theorem *[J. Hadamard, P. de la Vallée-Poussin 1896]*

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1$$

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2. Iteration and recursion
3. Visualization
4. Symbolic inversion
5. Pseudorandomness
6. Time, memory and precision
7. Linear algebra and graph theory

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Algorithm 2.1 Sieve of Eratosthenes

```
procedure SIEVE(N)
    Construct a list L = [2, ..., N]
    Let p = 2
    while p < N do
        Remove ip from L for each integer i ≥ 2
        Let p be the smallest element in L greater than p
    end while
return L                                  ▷ L contains all primes smaller than N
end procedure
```

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Question

What is

$$1.772453851?$$

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What is

$$1, 2, 5, 12, 32, 94, 289, 910?$$